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Bootstrap Percolation on Homogeneous Trees Has 2 Phase Transitions

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Abstract We study the threshold θ bootstrap percolation model on the homogeneous tree with degree $b+1$, $2 \leq \theta \leq b$, and initial density p. It is known that there exists a nontrivial critical value for p, which we call p_f, such that a) for $p > p_f$, the final bootstrapped configuration is fully occupied for almost every initial configuration, and b) if $p < p_f$, then for almost every initial configuration, the final bootstrapped configuration has density of occupied vertices less than 1. In this paper, we establish the existence of a distinct critical value for p, p_c, such that $0 < p_c < p_f$, with the following properties: 1) if $p \leq p_c$, then for almost every initial configuration there is no infinite cluster of occupied vertices in the final bootstrapped configuration; 2) if $p > p_c$, then for almost every initial configuration there are infinite clusters of occupied vertices in the final bootstrapped configuration. Moreover, we show that 3) for $p < p_c$, the distribution of the occupied cluster size in the final bootstrapped configuration has an exponential tail; 4) at $p = p_c$, the expected occupied cluster size in the final bootstrapped configuration is infinite; 5) the probability of percolation of occupied vertices in the final bootstrapped configuration is continuous on $[0, p_f]$ and analytic on (p_c, p_f), admitting an analytic continuation from the right at p_c and, only in the case $\theta = b$, also from the left at p_f.

Keywords Bootstrap percolation · Trees · Percolation · Phase transition · Exponential decay · Analiticity

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1 Introduction

Bootstrap percolation is a process of continued interest for physicists as well as mathematicians. For a review, we refer the reader to [1]. Here we obtain new results for the process on a homogeneous tree. We show that in addition to the well known critical point p_f, above which the tree becomes fully occupied, there is a distinct critical point p_c above which occupied sites percolate. We then prove several results that indicate the sharpness of the transition to percolation, and analyticity of the percolation probability between these critical points.

This paper was motivated in part by our work on the threshold θ contact process on homogeneous trees [6]. In the latter model, sites get infected by neighboring infected sites, provided there are at least θ of them, at rate λ, and recover unconditionally at rate 1. Bootstrap percolation corresponds in a heuristic sense to the $\lambda \to \infty$ limit of this model at time $0+$, and was used there as a tool in the study of the regime of large λ. Absence of percolation of 1's in the bootstrap percolation process implies that the threshold contact process converges to the state with all spins 0. We wonder if the presence of an intermediate phase for the bootstrap percolation process implies the existence of an intermediate phase also for the threshold contact process with large λ.

In regard to bootstrap percolation on the cubic lattices $\mathbb{Z}^d$, $d \geq 2$, one cannot hope for the same results that we have here, with $0 < p_c < p_f < 1$. This is so because for these models $p_f = 0$ or $p_f = 1$, according to whether $\theta \leq d$, or $\theta > d$, respectively, as proved in [11]. Of course, we have then $0 = p_c = p_f$ in the former case and $0 < p_c < p_f = 1$ in the latter case. There is nevertheless an important way in which one can describe a surrogate of a non-trivial transition point p_f for the models with $p_f = 0$, as pointed out originally in [2], and further studied in various papers, including [4] and [9]. For this purpose one takes a d-dimensional box of sidelength n and scales $n \to \infty$ at the same time as $p \to 0$. If the compromise between n and p is of the appropriate form (for instance $n = \exp(C/p)$ in the case $d = \theta = 2$), then as a parameter that mediates that relationship (the parameter C in this $d = \theta = 2$ case) is varied, one can either have the probability that the box becomes fully occupied converge to 0 or to 1. The mentioned parameter undergoes therefore a non-trivial transition. It is possible that in this way two distinct critical points may be produced, one corresponding to full occupancy and one corresponding to an analog in finite volume of percolation. For this purpose, consider the events E_f that the box is eventually fully occupied, and E_p that each pair of opposite faces of the box are eventually connected by a path of occupied sites. It is conceivable that in $d \geq 3$ (but not in $d = 2$), the way of scaling n with p to see a transition from $\mathbb{P}(E) \approx 0$ to $\mathbb{P}(E) \approx 1$ would depend on whether $E = E_f$ or $E = E_p$.

We turn now to the notation and definitions needed in this paper. Let $\mathbb{T}_b$ be the (unoriented) homogeneous tree with degree $b+1$, where $b \geq 2$, and let $\mathbb{V}_b$ be its vertex set. We also consider $\vec{\mathbb{T}}_b$, the oriented homogeneous tree with degree $b+1$; this is the graph with the same vertex set $\mathbb{V}_b$ as $\mathbb{T}_b$, and oriented edges such that incident to each vertex there are b outgoing edges and 1 incoming edge. For $x \in \mathbb{V}_b$, let $\mathcal{N}_x$ be the set of nearest neighbors of x in $\mathbb{T}_b$, that is, $y \in \mathbb{V}_b$ incident to which there are edges of $\mathbb{T}_b$ which are incident to x as well. We also define $\vec{\mathcal{N}}_x$ as the set of oriented nearest neighbors of x in $\vec{\mathbb{T}}_b$, that is, the $y \in \mathbb{V}_b$ incident to which there are *outgoing* edges of $\vec{\mathbb{T}}_b$ from x. We will fix an arbitrary vertex of $\mathbb{V}_b$ as the root of $\mathbb{T}_b$, and denote it R. We will use the shorthand $\mathcal{N} = \mathcal{N}_R$ and $\vec{\mathcal{N}} = \vec{\mathcal{N}}_R$. For $x \in \mathbb{V}_b$ we will denote by x^- the unique element of $\mathcal{N}_x \setminus \vec{\mathcal{N}}_x$; we will also consider the *forward trees from* x, $\mathbb{T}_b^{+,x}$ and $\vec{\mathbb{T}}_b^{+,x}$, consisting respectively of the connected components containing x of the subgraphs of $\mathbb{T}_b$ and $\vec{\mathbb{T}}_b$ obtained by removing x^- along with all edges of $\mathbb{T}_b$ and $\vec{\mathbb{T}}_b$ incident on x^-; x will be called the root of the respective trees. We will write

$\mathbb{T}_b^+ = \mathbb{T}_b^{+,R}$ and $\vec{\mathbb{T}}_b^+ = \vec{\mathbb{T}}_b^{+,R}$ for short. Let also $\mathbb{V}_b^{+,x}$ denote the common vertex set of $\mathbb{T}_b^{+,x}$ and $\vec{\mathbb{T}}_b^{+,x}$, with the shorthand notation $\mathbb{V}_b^+$ for $\mathbb{V}_b^{+,R}$.

Below we will consider elements of $\{0,1\}^\Lambda$, with Λ a subset of $\mathbb{V}_b$. For η a given such element, which we call *configuration*, and $x \in \Lambda$, we say that x is *occupied* (in η) if $\eta(x) = 1$; and *vacant*, otherwise. We will say that a subset Λ' of Λ is occupied (resp. vacant) if $\eta(x) = 1$ (resp. 0) for all $x \in \Lambda'$.

We now define the bootstrap percolation model with threshold θ, an integer such that $2 \le \theta \le b$, and initial density p on $\mathbb{V}_b$. See [3] and references therein for background on those models. Let the initial configuration $\eta_0 \in \{0,1\}^{\mathbb{V}_b}$ be chosen according to a product of Bernoullis with parameter p. And for $n \ge 1$ and arbitrary $x \in \mathbb{V}_b$ set:

$$\eta_n(x) = \begin{cases} 1, & \text{if } \eta_{n-1}(x) = 1; \\ 1, & \text{if } \eta_{n-1}(x) = 0 \text{ and } \sum_{y \in \mathcal{N}_x} \eta_{n-1}(x) \ge \theta; \\ 0, & \text{if } \eta_{n-1}(x) = 0 \text{ and } \sum_{y \in \mathcal{N}_x} \eta_{n-1}(x) < \theta. \end{cases} \tag{1.1}$$

We note that η_n is nondecreasing in n and thus

$$\lim_{n\to\infty} \eta_n =: \eta_\infty \tag{1.2}$$

is well defined. We call η_∞ *final (bootstrapped) configuration*. We will also call $(\eta_n)_{n\ge 0}$ the *(unoriented) bootstrapping dynamics*.

Similarly we consider the oriented model $\vec{\eta}_0 = \eta_0$ and $\vec{\eta}_n$ defined recursively as in (1.1), with $\vec{\eta}_{n-1}$ replacing η_{n-1}, and $\vec{\mathcal{N}}_x$ replacing $\mathcal{N}_x$. Monotonicity also gives sense to

$$\lim_{n\to\infty} \vec{\eta}_n =: \vec{\eta}_\infty, \tag{1.3}$$

the final configuration of the oriented model.

The only randomness entering these models is in the initial configuration η_0. Let $\mathbb{P}_p$ denote the underlying probability measure, and $\mathbb{E}_p$ the corresponding expectation. For $n = 0, 1, \dots, \infty$, let us define

$$\vec{p}_n = \mathbb{E}_p(\vec{\eta}_n(R)), \qquad p_n = \mathbb{E}_p(\eta_n(R)). \tag{1.4}$$

Due to the translation invariance of $\mathbb{P}_p$ and of the dynamical rules, we have that the distributions of $\vec{\eta}_n$ and η_n are translation invariant for every $n = 0, 1, \dots, \infty$, so the particular choice of root is not important in (1.4). It is well known [3] that if we set

$$p_f = \inf\{p \in [0,1] : \mathbb{P}_p(\eta_\infty \equiv 1) = 1\} = \inf\{p \in [0,1] : p_\infty = 1\} \tag{1.5}$$

$$\vec{p}_f = \inf\{p \in [0,1] : \mathbb{P}_p(\vec{\eta}_\infty \equiv 1) = 1\} = \inf\{p \in [0,1] : \vec{p}_\infty = 1\} \tag{1.6}$$

then

$$p_f = \vec{p}_f \in (0,1). \tag{1.7}$$

For $p \in [0, p_f]$ it is interesting to study the properties of the random configurations η_∞ and $\vec{\eta}_\infty$. Here we will study percolation of these configurations.

Given $\eta_0 \in \{0,1\}^{\mathbb{V}_b}$, for $x \in \mathbb{V}_b$, let $\mathcal{C}_x$ (resp. $\vec{\mathcal{C}}_x$) denote the cluster of occupied vertices of $\mathbb{T}_b$ (resp. $\vec{\mathbb{T}}_b$) containing vertex x in the final configuration η_∞ (resp. $\vec{\eta}_\infty$). Namely $\mathcal{C}_x$ (resp. $\vec{\mathcal{C}}_x$) is the maximal set of occupied vertices y of $\mathbb{T}_b$ in η_∞ (resp. $\vec{\eta}_\infty$) such that there is a finite path in $\mathcal{C}_x$ (resp. $\vec{\mathcal{C}}_x$) connecting x to y, where by a path we mean an ordered collection $\{y_1, y_2, \ldots\} \subset \mathbb{V}_b$ such that $y_{i+1} \in \mathcal{N}_{y_i}$ (resp. $\vec{\mathcal{N}}_{y_i}$) for $i \geq 1$. We will denote for short $\mathcal{C} = \mathcal{C}_R$ and $\vec{\mathcal{C}} = \vec{\mathcal{C}}_R$.

We say that there is *percolation* at x in η_∞ (resp. in $\vec{\eta}_\infty$) if $|\mathcal{C}_x| = \infty$ (resp. $|\vec{\mathcal{C}}_x| = \infty$). We define now the percolation critical points:

$$p_c = \inf\{p \in [0,1] : \mathbb{P}_p(|\mathcal{C}| = \infty) > 0\} \tag{1.8}$$

$$\vec{p}_c = \inf\{p \in [0,1] : \mathbb{P}_p(|\vec{\mathcal{C}}| = \infty) > 0\}. \tag{1.9}$$

We state the main result of this paper next for the unoriented model alone.

Theorem 1.1

1.

$$0 < p_c < p_f \tag{1.10}$$

2. *For $p < p_c$, there exist positive finite constants c_1, c_2 such that for all $k \geq 0$*

$$\mathbb{P}_p(|\mathcal{C}| > k) \leq c_1 e^{-c_2 k}. \tag{1.11}$$

3. *At $p = p_c$,*

$$\mathbb{P}_p(|\mathcal{C}| > k) \to 0, \text{ as } k \to \infty; \tag{1.12}$$

$$\mathbb{E}_p(|\mathcal{C}|) = \infty. \tag{1.13}$$

4. *For $p > p_c$, there exist positive finite constants c_3, c_4 such that for all $k \geq 0$*

$$\mathbb{P}_p(k < |\mathcal{C}| < \infty) \leq c_3 e^{-c_4 k}. \tag{1.14}$$

5. $\pi(p) := \mathbb{P}_p(|\mathcal{C}| = \infty)$ *is continuous on* $[0, p_f]$, *analytic on* (p_c, p_f), *and admits an analytic continuation from the right at p_c. If $\theta = b$, then π is continuous on* $[0,1]$.

The smoothness properties of π on the left of p_f depend on b and θ, as stated in the following result.

Theorem 1.2

1. *If $\theta = b$, then π admits an analytic continuation from the left at p_f.*
2. *If $\theta < b$, then as $p \uparrow p_f$*

$$\frac{d}{dp}\pi(p) \to \infty. \tag{1.15}$$

It will become clear in the arguments used to prove the above results that oriented versions of them hold as well. Moreover, in contrast to (1.7),

$$p_c < \vec{p}_c.$$

It is interesting to note that when $\mathbb{P}_p(\eta_\infty \equiv 1) = 0$, vacant sites must percolate in η_∞ (since finite clusters of vacant sites are eliminated by the dynamics). Therefore, in the intermediate phase between p_c and p_f infinite clusters of vacant and of occupied sites coexist.

It is interesting to observe that for some values of b and θ, the infinite clusters of occupied sites that occur in the intermediate regime between p_c and p_f are not present at time 0, and are therefore produced by the dynamics. This is the case, for instance, when $\theta = 2$ and b is large. We know from [3] that p_f is then of order $1/b^2$. But the critical point for percolation at time 0 is $1/b \gg 1/b^2$. The existence of the intermediate phase then shows that sometimes the bootstrap percolation dynamics is "strong enough to create infinite clusters", but "not strong enough to make all sites occupied".

Our proof of Theorem 1.1 requires us to use several tools that are known to imply (1.7). Because those tools are somewhat different than those found in [3] and other papers, we review them in Sect. 2. In Sects. 3 and 4 we analyze the oriented and unoriented models, respectively. An Appendix collects auxiliary results and supplementary proofs.

2 Full Occupancy

The results in this section are well known. We nevertheless choose to present proofs for them since we found no clear cut reference for each of them specifically.

Proposition 2.1 *$\vec{p}_\infty$ is the smallest solution in* $[0, 1]$ *of*

$$x = f_p(x), \tag{2.1}$$

where $f_p(x) = p + q \sum_{k=\theta}^{b} \binom{b}{k} x^k (1-x)^{b-k}$, with $q = 1 - p$.

Remark 2.2 1 is always a solution of (2.1). One readily checks that for p close enough to 1, this is the only solution, and for p close enough to 0, there are smaller solutions in $[0, 1]$.

Proof of Proposition 2.1 The key observation is that for every $n \geq 0$, the random variables $\{\vec{\eta}_n(x);\ x \in \vec{\mathcal{N}}\}$ are independent Bernoullis with common parameter $\vec{p}_n$, and are independent of $\vec{\eta}_0(R)$. From the dynamical rules, we have that, for $n \geq 1$, $\vec{\eta}_n(R) = 1$ if and only if either $\vec{\eta}_0(R) = 1$ or $\vec{\eta}_0(R) = 0$ and $\sum_{x \in \vec{\mathcal{N}}} \vec{\eta}_{n-1}(x) \geq \theta$. By the latter part of the key observation above, we conclude that

$$\begin{aligned} \vec{p}_n &= p_0 + (1 - p_0)\mathbb{P}_p\left(\sum_{x \in \vec{\mathcal{N}}} \vec{\eta}_{n-1}(x) \geq \theta \,\middle|\, \vec{\eta}_0(R) = 0 \right) \\ &= p + q\mathbb{P}_p\left(\sum_{x \in \vec{\mathcal{N}}} \vec{\eta}_{n-1}(x) \geq \theta \right) = f_p(\vec{p}_{n-1}), \end{aligned}$$

where in the latter passage, we have used the first part of the key observation above, from which follows that $\sum_{x \in \vec{\mathcal{N}}} \vec{\eta}_{n-1}(x)$ has a binomial distribution with b trials and probability of success $\vec{p}_{n-1}$ in each trial. From the monotonicity of $\vec{p}_n$ in n, the continuity and increasing monotonicity of $f_p(x)$ in x and the that fact that $\vec{p}_\infty = \lim_{n \to \infty} \vec{p}_n$, the result follows. □

Corollary 2.3

$$\vec{p}_f = \sup\{p \in [0, 1] :\ (2.1) \textit{ has a solution in } (0, 1)\} \tag{2.2}$$

Remark 2.4 From obvious properties of $f_p(x)$, we have that, for $\bar{p} = \vec{p}_f$ and $\bar{p}_\infty = \vec{p}_\infty(\bar{p})$, $f_{\bar{p}}'(\bar{p}_\infty) = 1$, where $f_{\bar{p}}'$ is the derivative of $f_{\bar{p}}$.

Proposition 2.5 *We have*

$$p_\infty = p + q \sum_{k=\theta}^{b+1} \binom{b+1}{k} \vec{p}_\infty^{\,k} (1-\vec{p}_\infty)^{b+1-k}. \tag{2.3}$$

Proof Let $x_1, \dots, x_{b+1}$ be an enumeration of $\mathcal{N}_R$, and consider unoriented bootstrap percolation models on $\mathbb{T}^{+,x_i}$, $i = 1, \dots, b+1$, (defined in the obvious way, and started from η_0 restricted to the respective subgraph). Let $\zeta_n^{(i)}$, $n = 0, 1, \dots, \infty$ denote the successive configurations of the unoriented bootstrap percolation models on $\mathbb{T}^{+,x_i}$, $i = 1, \dots, b+1$. Now, on $\{\eta_0(R) = 0\}$ we have that $\eta_\infty(R) = 1$ iff $\sum_{i=1}^{b+1} \zeta_\infty^{(i)}(x_i) \geq \theta$. Since $\zeta_\infty^{(i)}$, $i = 1, \dots, b+1$, are i.i.d., we conclude that (2.3) holds with $\vec{p}_\infty$ replaced by $\mathbb{E}_p(\zeta_\infty^{(1)})$.

Consider now oriented bootstrap percolation on $\vec{\mathbb{T}}^{+,x_1}$. Notice that it is identical to oriented bootstrap percolation on $\vec{\mathbb{T}}_b$ restricted to $\vec{\mathbb{T}}^{+,x_1}$. Let $\vec{\zeta}_n^{(1)}$, $n = 0, 1, \dots, \infty$ denote the successive configurations of the former model. We recall that $\zeta_0^{(1)} = \vec{\zeta}_0^{(1)} = \eta_0$ restricted to $\mathbb{V}_b^{+,x_1}$.

To finish the proof, we claim that if $\vec{\zeta}_\infty^{(1)}(x_1) = 0$, then $\zeta_\infty^{(1)}(x_1) = 0$. We introduce a piece of terminology before proceeding; we will say that a vertex $x \in \mathbb{V}_b^{+,x_1}$ is *protected* if $\vec{\zeta}_\infty^{(1)}(x) = 0$. Let us also denote $T_1 = T_1^{(b)} := \mathbb{T}^{+,x_1}$. To argue the claim, we start by observing that if x_1 is protected, then there must be a vacant subtree of $T_1^{(b)}$, denoted $\mathcal{T}$, with x_1 as root, which is isomorphic to $T_1^{(b-\theta+1)}$. This follows from the fact that in order that $x \in \vec{T}_1$ be protected, we must have $\vec{\zeta}_0^{(1)}(x) = 0$ and at least $b - \theta + 1$ protected vertices in $\vec{\mathcal{N}}_x$; for each such x, let $\mathcal{P}_x$ be a(n arbitrary) choice of exactly $b - \theta + 1$ such protected vertices. Then making $\mathcal{T}_0 = \{x_1\}$ and, for $n \geq 1$, $\mathcal{T}_n = \bigcup_{x \in \mathcal{T}_{n-1}} \mathcal{P}_x$, we will have that $\bigcup_n \mathcal{T}_n$ may be taken as $\mathcal{T}$. Now $\mathcal{T}$ is invariant under the unoriented threshold θ bootstrap percolation dynamics on T_1, since every vertex in it is vacant and has fewer than θ occupied nearest neighbor vertices in T_1. Thus $\zeta_\infty^{(1)}(x_1) = 0$. □

It follows from Proposition 2.5 that the critical parameters in (1.5–1.6) are actually equal.

Corollary 2.6

$$\vec{p}_f = p_f \tag{2.4}$$

Proof It is clear from (2.3) that $p_\infty = 1$ iff $\vec{p}_\infty = 1$. □

Remark 2.7 An obvious corollary of the above, namely that p_f equals the right hand side of (2.2), is the main statement of Proposition 1.2 of [3].

Remark 2.8 One readily checks from the above that $0 < p_f < 1$. For $p < 1$, the function $f_p(x)$ is strictly increasing and analytic; thus for $p < p_f$ its derivative at $\vec{p}_\infty$ is strictly less than 1. It follows that the derivative at $\vec{p}_\infty$ of $f_p(x) - x$ is nonzero. We can then invoke the Analytic Implicit Function Theorem to conclude that $\vec{p}_\infty = \vec{p}_\infty(p)$ and $p_\infty = p_\infty(p)$ are analytic in $[0, p_f)$. From the obvious increasing monotonicity of $\vec{p}_\infty$ and p_∞ we conclude that they are both strictly increasing in $[0, p_f]$.

p_f can be readily explicitly computed in cases $\theta = 2$ and b; in the latter case, $p_f = 1 - 1/b$. See Proposition 1.2 of [3]. It can also be readily checked that $\vec{p}_\infty$ and p_∞ are continuous at p_f for $\theta = b$ and discontinuous (but left continuous) at p_f for $\theta < b$.

3 The Oriented Case

We will describe $\vec{\mathcal{C}}$, the occupied cluster of the root in $\vec{\eta}_\infty$, as *cluster of clusters* of branching processes, which we now discuss.

We start by classifying an initially vacant vertex as *weakly vacant*, if it is eventually occupied by the (oriented) dynamics, and *strongly vacant*, if it is never occupied by that dynamics. That is, $x \in \mathbb{V}_b$ such that $\vec{\eta}_0(x) = \eta_0(x) = 0$ is weakly vacant if $\vec{\eta}_\infty(x) = 1$, and strongly vacant if $\vec{\eta}_\infty(x) = 0$. We will consider configurations of occupied, weakly vacant and strongly vacant vertices, $\xi \in \{1, \underline{0}, \overline{0}\}^{\mathbb{V}_b}$ as follows. For $x \in \mathbb{V}_b$, set

$$\xi(x) = \begin{cases} 1, & \text{if } \vec{\eta}_0(x) = 1, \\ \underline{0}, & \text{if } \vec{\eta}_0(x) = 0 \text{ but } \vec{\eta}_\infty(x) = 1, \\ \overline{0}, & \text{if } \vec{\eta}_\infty(x) = 0. \end{cases} \tag{3.1}$$

We will consider next the cluster $\mathcal{W}_x$ of weakly vacant vertices, or $\underline{0}$-vertices, containing a given vertex x. That is,

$$\mathcal{W}_x = \{y \in \mathbb{V}_b : \text{there exist } x = x_0, x_1, \ldots, x_n = y \text{ with } x_i \in \vec{\mathcal{N}}_{x_{i-1}},$$
$$i = 1, \ldots n, \text{ and } \xi(x_i) = \underline{0}, i = 0, \ldots n\}.$$

$\mathcal{W}_x$ will be empty, if $\xi(x) \neq \underline{0}$. Otherwise, it is a cluster of a branching process. To argue that, we start by letting, for given $x \in \mathbb{V}_b$, O_x, W_x, S_x denote respectively the numbers of initially 1-, $\underline{0}$-, and $\overline{0}$-neighbors of x; that is,

$$(O_x, W_x, S_x) = \left(\sum_{y \in \vec{\mathcal{N}}_x} 1\{\xi(y) = 1\}, \sum_{y \in \vec{\mathcal{N}}_x} 1\{\xi(y) = \underline{0}\}, \sum_{y \in \vec{\mathcal{N}}_x} 1\{\xi(y) = \overline{0}\} \right). \tag{3.2}$$

Remark 3.1 The independence of $\{\xi(y), y \in \vec{\mathcal{N}}_x\}$ implies that (O_x, W_x, S_x) is trinomial with parameters b (number of trials) and $p, \vec{r}_\infty, \vec{q}_\infty$ (probabilities of resp. $1, \underline{0}, \overline{0}$), where $\vec{r}_\infty = \vec{p}_\infty - p$ and $\vec{q}_\infty = 1 - \vec{p}_\infty$. Also, $\xi(x) = \underline{0}$ if and only if $\vec{\eta}_0(x) = 0$ and $S_x \leq b - \theta$; thus, the conditional distribution of W_x given $\xi(x) = \underline{0}$ is the same as that of W_x given $S_x \leq b - \theta$. We conclude that $\mathcal{W}_x$ either is empty, with probability $1 - \vec{r}_\infty$, or, with probability $\vec{r}_\infty$, it is the cluster of a branching process initiated with one individual and with offspring distribution given by the conditional distribution of W_x given $S_x \leq b - \theta$.

We now introduce the *local cluster* $\mathcal{L}_x$ of occupied vertices of $\vec{\eta}_\infty$ containing x, and its boundary $\mathcal{O}_x$ of initially occupied vertices.

If $\xi(x) = \overline{0}$, let

$$\mathcal{L}_x = \mathcal{O}_x = \emptyset. \tag{3.3}$$

If $\xi(x) = 1$, let

$$\mathcal{L}_x = \mathcal{O}_x = \{x\}. \tag{3.4}$$

If $\xi(x) = \underline{0}$, let

$$\mathcal{L}_x = \mathcal{W}_x \cup \mathcal{O}_x, \quad \text{with } \mathcal{O}_x = \{y \in \bar{\partial}\mathcal{W}_x : \xi(y) = 1\}, \tag{3.5}$$

where, given a nonempty subset Λ of $\mathbb{V}_b$, $\bar{\partial}\Lambda = \{y \notin \Lambda : y \in \vec{\mathcal{N}}_z \text{ for some } z \in \Lambda\}$ is the oriented outer boundary of Λ.

Remark 3.2 $\vec{\mathcal{C}}_x$ can be obtained from local clusters and their 1-boundaries by the following iteration. Let $C_0 = \mathcal{L}_x$, $O_0 = \mathcal{O}_x$, and for $n \geq 0$ make

$$C_{n+1} = \bigcup_{y \in O_n} \bigcup_{z \in \vec{\mathcal{N}}_y} \mathcal{L}_z; \qquad O_{n+1} = \bigcup_{y \in O_n} \bigcup_{z \in \vec{\mathcal{N}}_y} \mathcal{O}_z. \tag{3.6}$$

Then

$$\vec{\mathcal{C}}_x = \bigcup_{n \geq 0} C_n. \tag{3.7}$$

We now notice that, for every $n \geq 0$, with $O_n \neq \emptyset$, the portion of the ξ-configuration forward from O_n is independent from that restricted to $\bigcup_{i=0}^n C_i$. From this and the above construction, it follows that $O := (O_n)_{n \geq 0}$ are the successive generations of a branching process with initial distribution given by a copy of $|\mathcal{O}|$, and offspring distribution given by the sum of b i.i.d. copies of $|\mathcal{O}|$, independent of the initial distribution, where $\mathcal{O} = \mathcal{O}_R$, and that almost surely $|\vec{\mathcal{C}}| = \infty$ if and only if that branching process survives.

The following sequence of lemmas and propositions exploits Remark 3.2.

Lemma 3.3 *Let* $\nu = \nu(p) = \mathbb{E}_p(|\mathcal{O}|)$ *and*

$$M = M(p) = q \sum_{k=\theta}^{b} k \binom{b}{k} \vec{p}_\infty^{\,k-1} \vec{q}_\infty^{\,b-k}. \tag{3.8}$$

Then either $M < 1$ *and* $\nu = \frac{p}{1-M}$ *or else* $\nu = \infty$.

Remark 3.4 $M(0) = 0$ and by Remark 2.8, we have that M is analytic in $[0, p_f)$ and left continuous at p_f.

Lemma 3.5 *Set* $\tilde{p} = \inf\{p \in [0, p_f] : M(p) = 1\}$. *Then*

$$\tilde{p} \begin{cases} < p_f, & \text{if } 2 \leq \theta < b, \\ = p_f, & \text{if } 2 \leq \theta = b. \end{cases} \tag{3.9}$$

Remark 3.6 We will see below that $M(p) = 1$ has at least 1 solution in $[0, p_f]$. By the continuity of M we then have that $\tilde{p}$ is the smallest such solution.

Remark 3.7 By the above results and standard facts about analytic function theory, we have that, on $[0, \tilde{p})$, $\nu = \frac{p}{1-M}$ is analytic. Note also that $\nu(0) = 0$ and $\nu(p) \to \infty$ as $p \uparrow \tilde{p}$.

Lemma 3.8 *For* $p \in [0, \tilde{p})$ *there exist positive finite constants* c_1, c_2 *such that for all* $k \geq 0$

$$\mathbb{P}_p(|\mathcal{L}| > k) \leq c_1 e^{-c_2 k}, \tag{3.10}$$

where $\mathcal{L} = \mathcal{L}_R$.

Warning: The constants c_1, c_2 in the above lemma are not necessarily the same as those in the second part of Theorem 1.1. Throughout, constants denoted c_i, $i \geq 0$, may be different in different appearances.

Proposition 3.9 *We have*

$$\vec{p}_c = \inf\{p \in [0, p_f] : \nu(p) = 1/b\} \in (0, \tilde{p}). \tag{3.11}$$

For $p < \vec{p}_c$ there exist positive finite constants c_1, c_2 such that for all $k \geq 0$

$$\mathbb{P}_p(|\vec{\mathcal{C}}| > k) \leq c_1 e^{-c_2 k}. \tag{3.12}$$

And $\mathbb{P}_{\vec{p}_c}(|\vec{\mathcal{C}}| = \infty) = 0$, $\mathbb{E}_{\vec{p}_c}(|\vec{\mathcal{C}}|) = \infty$.

Proof of Lemma 3.3 Suppose that $\nu < \infty$. Now consider the events

$$A = \left\{ \sum_{x \in \vec{\mathcal{N}}} 1\{\xi(x) = 1 \text{ or } \underline{0}\} \geq \theta \right\} \tag{3.13}$$

and, for $k \leq b$ and a given selection of k vertices $x_1, \ldots x_k$ of $\vec{\mathcal{N}}$,

$$A_k = \left\{ \xi(x_i) = 1 \text{ or } \underline{0} \text{ for } i = 1, \ldots, k \right\}. \tag{3.14}$$

Then

$$\begin{aligned}
\nu = \mathbb{E}_p(|\mathcal{O}|) &= p + q\mathbb{E}_p\left(\sum_{x \in \vec{\mathcal{N}}} |\mathcal{O}_x|; A \right) \\
&= p + q \sum_{k=\theta}^{b} \binom{b}{k} \mathbb{E}_p \left(|\mathcal{O}_{x_1}| + \cdots + |\mathcal{O}_{x_k}|; A_k \right) \\
&= p + q \sum_{k=\theta}^{b} k \binom{b}{k} \mathbb{E}_p \left(|\mathcal{O}_{x_1}|; A_k \right) \\
&= p + q \sum_{k=\theta}^{b} k \binom{b}{k} \mathbb{E}_p \left(|\mathcal{O}_{x_1}|; \xi(x_1) = 1 \text{ or } \underline{0} \right) \mathbb{P}_p(\xi(x_2) = 1 \text{ or } \underline{0}) \\
&\quad \times \cdots \mathbb{P}_p(\xi(x_k) = 1 \text{ or } \underline{0}) \mathbb{P}_p(\xi(x_{k+1}) = \bar{0}) \cdots \mathbb{P}_p(\xi(x_b) = \bar{0}) \\
&= p + q \sum_{k=\theta}^{b} k \binom{b}{k} \nu \vec{p}_\infty^{\,k-1} \vec{q}_\infty^{\,b-k} = p + \nu M,
\end{aligned}$$

where $\{x_{k+1}, \ldots, x_b\} = \vec{\mathcal{N}} \setminus \{x_1, \ldots, x_k\}$. The result follows. □

Proof of Lemma 3.5 When $2 \leq \theta = b$, we have by Remark 2.4 and (2.4) that at $p = \bar{p} = p_f$,

$$1 = f_p'(\bar{p}_\infty) = qb\bar{p}_\infty^{\,b-1} = M(p).$$

Suppose now that $p < p_f = 1 - 1/b$ (see Remark 2.8) and $M(p) = qb\vec{p}_\infty^{\,b-1} = 1$. It follows that $\vec{p}_\infty = (qb)^{-1/(b-1)}$. Now $\vec{p}_\infty$ also satisfies (2.1), which in this case, since from the strict increasing monotonicity of $\vec{p}_\infty$ in $[0, p_f]$ (see Remark 2.8) we have $\vec{p}_\infty(p) < \vec{p}_\infty(p_f) = 1$, is equivalent to $x + x^2 + \cdots + x^{b-1} = p/q$. Thus

$$\frac{p}{q} = \vec{p}_\infty + \cdots + \vec{p}_\infty^{\,b-1} = (qb)^{-\frac{1}{b-1}} + \cdots + (qb)^{-1} \geq \frac{b-1}{qb},$$

since $bq \geq 1$. It follows that $p \geq 1 - 1/b$, in contradiction with the hypothesis; we conclude that $\tilde{p} = p_f$.

When $2 \leq \theta < b$, at $p = \bar{p} = p_f$, again by Remark 2.4 and (2.4) we have that

$$
\begin{aligned}
1 &= f_p'(\bar{p}_\infty) \\
&= q \frac{d}{dx} \left\{ \sum_{k=\theta}^{b} \binom{b}{k} x^k (1-x)^{b-k} \right\}_{x=\bar{p}_\infty} \\
&= q \sum_{k=\theta}^{b} k \binom{b}{k} \bar{p}_\infty^{k-1} \bar{q}_\infty^{b-k} - q \sum_{k=\theta}^{b-1} (b-k) \binom{b}{k} \bar{p}_\infty^{k} \bar{q}_\infty^{b-k-1} \\
&= M(p) - q \sum_{k=\theta}^{b-1} (b-k) \binom{b}{k} \bar{p}_\infty^{k} \bar{q}_\infty^{b-k-1} \\
&< M(p),
\end{aligned}
$$

since $q \sum_{k=\theta}^{b-1} (b-k) \binom{b}{k} \bar{p}_\infty^{k} \bar{q}_\infty^{b-k-1} > 0$ in this case. So $M(p) > 1$, and since $M(0) = 0$ and M is continuous, it follows that $\tilde{p} < p = p_f$. □

Proof of Lemma 3.8 We claim that M is the offspring mean of the branching process involved in $\mathcal{W} := \mathcal{W}_R$ (see Remark 3.1). Indeed, that mean equals

$$
\begin{aligned}
&\mathbb{E}_p(W | S \leq b - \theta) \\
&\quad = \frac{1}{\mathbb{P}_p(S \leq b-\theta)} \sum_{s=0}^{b-\theta} \sum_{w=0}^{b-s} w \mathbb{P}_p(W = w, S = s) \\
&\quad = \frac{1}{\mathbb{P}_p(S \leq b-\theta)} \sum_{s=0}^{b-\theta} \sum_{w=1}^{b-s} w \frac{b!}{w!s!(b-w-s)!} \vec{r}_\infty^{\,w} \vec{q}_\infty^{\,s} p^{b-w-s} \\
&\quad = \frac{b\vec{r}_\infty}{\mathbb{P}_p(S \leq b-\theta)} \sum_{s=0}^{b-\theta} \sum_{w=1}^{b-s} \frac{(b-1)!}{(w-1)!s!(b-w-s)!} \vec{r}_\infty^{\,w-1} \vec{q}_\infty^{\,s} p^{b-w-s} \\
&\quad = \frac{b\vec{r}_\infty}{\mathbb{P}_p(S \leq b-\theta)} \sum_{s=0}^{b-\theta} \binom{b-1}{s} \vec{p}_\infty^{\,b-1-s} \vec{q}_\infty^{\,s} \\
&\quad = \frac{\vec{r}_\infty M(p)}{q \mathbb{P}_p(S \leq b-\theta)} = \frac{\mathbb{P}_p(\xi(R) = \underline{0})}{q \mathbb{P}_p(S \leq b-\theta)} M(p) = M(p),
\end{aligned}
$$

where $S = S_R$ and $W = W_R$, and the claim is justified.

Thus for $p < \tilde{p}$, $\mathcal{W}$ is subcritical, and since its offspring distribution is bounded, Proposition A.1 applies and we get the exponential decay of the distribution of $|\mathcal{W}|$. The result now follows from (3.5) and the obvious bound $|\mathcal{O}| \leq b|\mathcal{W}| + 1$. □

Proof of Proposition 3.9 First note that from Remark 3.2,

$$
b\nu(p) > 1 \quad \Leftrightarrow \quad \mathbb{P}_p(|\vec{\mathcal{C}}| = \infty) > 0. \tag{3.15}
$$

Also, by the exponential decay of $|\mathcal{O}|$ at $p < \tilde{p}$ (which follows by (3.3–3.5) and Lemma 3.8), we may invoke Theorem I.13.1 in [8] to get that

$$b\nu(p) = 1 \quad \Rightarrow \quad \mathbb{E}_p(|\vec{\mathcal{C}}|) = \infty. \tag{3.16}$$

We also have

$$b\nu(p) = 1 \quad \Rightarrow \quad \mathbb{P}_p(|\vec{\mathcal{C}}| = \infty) = 0. \tag{3.17}$$

In contrast to (3.2–3.16), we claim that the same Remark 3.2 and Lemma 3.8 yield

$$b\nu(p) < 1, \quad p < \tilde{p} \quad \Rightarrow \quad \mathbb{P}_p(|\vec{\mathcal{C}}| > k) \le c_1 e^{-c_2 k}, \quad c_1, c_2 \in (0, \infty). \tag{3.18}$$

Indeed, the exponential decay of the offspring distribution of the O-branching process follows from (3.10); since it is subcritical, Remark A.3 applies and we get the exponential decay of the distribution of Z, the total size of its family. Now we get from Lemma A.4 that $|\vec{\mathcal{C}}| \le 2Z$ (see also Remark A.5). Equation (3.12) follows.

Set now

$$p' = \inf\{p \in [0, p_f] : b\nu(p) = 1\}.$$

From Lemma 3.5 and Remark 3.7, we learn that $0 < p' < \tilde{p} \le p_f$, $b\nu(p') = 1$ and that there are values of $p > p'$, arbitrarily close to p' such that $b\nu(p) \ne 1$. Therefore, from (3.16), we learn that $\mathbb{E}_{p'}(|\vec{\mathcal{C}}|) = \infty$, and by monotonicity in p, also $\mathbb{E}_p(|\vec{\mathcal{C}}|) = \infty$ for $p \ge p'$. From (3.18), we now know that $b\nu(p) \ge 1$ for $p > p'$. Hence, there are values of $p > p'$ arbitrarily close to p' such that $b\nu(p) > 1$. From (3.15), we have then $\vec{p}_c \le p'$. Using (3.18) again, we see that $\vec{p}_c \ge p'$, completing the proof. □

4 The Unoriented Case

We introduce now two modified oriented systems, out of which $\mathcal{C}$ will be constructed. For that we will consider another family of oriented trees. Say that a vertex x points to R in $\vec{\mathbb{T}}_b$ if there is an oriented path in $\vec{\mathbb{T}}_b$ starting at x and ending at R. And say that an edge points to R in case it starts at a vertex that points to R. Let $\vec{\mathbb{T}}_b^*$ be obtained from $\vec{\mathbb{T}}_b$ by reversing the orientation of the edges pointing to R. Let $\vec{\mathcal{N}}_x^*$ be the set of oriented nearest neighbors of x in $\vec{\mathbb{T}}_b^*$, with the shorthand $\vec{\mathcal{N}}^* = \vec{\mathcal{N}}_R^*$. Then $|\vec{\mathcal{N}}^*| = b + 1$ and $|\vec{\mathcal{N}}_x^*| = b$ for $x \in \mathbb{V}_b \setminus \{R\}$. For $x \ne R$, let $x^{*,-}$ be the unique element of $\mathcal{N}_x \setminus \vec{\mathcal{N}}_x^*$, and let $\vec{\mathbb{T}}_b^{*,+,x}$ be the graph consisting of the connected component containing x of the subgraph of $\vec{\mathbb{T}}_b^*$ obtained by removing $x^{*,-}$ along with all edges of $\vec{\mathbb{T}}_b^*$ incident on $x^{*,-}$. Let $\mathbb{V}_b^{*,+,x}$ denote the vertex set of $\vec{\mathbb{T}}_b^{*,+,x}$.

Let now $(\eta_n^*, n \ge 0)$ denote the oriented bootstrap dynamics in $\vec{\mathbb{T}}_b^*$ and let $\mathcal{C}^*$ denote the cluster of occupied vertices of η_∞^* containing R. Secondly, for $x \ne R$, let $(\eta_n^{+,x}, n \ge 0)$ denote the oriented bootstrap dynamics in $\vec{\mathbb{T}}_b^{*,+,x}$ with the following *perturbation*: the threshold at x is $\theta - 1$ instead of θ, which remains the threshold of all the remainder vertices of $\vec{\mathbb{T}}_b^{*,+,x}$; $\eta_0^{+,x}$ is the restriction of η_0 to the vertex set of $\vec{\mathbb{T}}_b^{*,+,x}$. For $y \in \mathbb{V}_b^{*,+,x}$, let $\mathcal{C}_y^{+,x}$ denote the cluster of occupied vertices of $\eta_\infty^{+,x}$ containing y, with $\mathcal{C}_x^+ := \mathcal{C}_x^{+,x}$ and $\mathcal{C}^+ = \mathcal{C}_{x_0}^+$, with x_0 a fixed element of $\vec{\mathcal{N}}_R$. By the spatial homogeneity of the initial condition and dynamical rules, the distribution of $|\mathcal{C}_x^+|$ is independent of $x \ne R$.

When $p \in [0, \vec{p}_c]$, it readily follows from Proposition 3.9 that, when $p < \vec{p}_c$, $\mathcal{C}^*$ and $\mathcal{C}^+$ are finite almost surely and that the distributions of $|\mathcal{C}^*|$ and $|\mathcal{C}^+|$ have exponentially decaying tails. Let now $X := |\bar{\partial}\mathcal{C}^+|$ and $\tilde{X} := |\tilde{\partial}\mathcal{C}^*|$, where $\tilde{\partial}\mathcal{C}^* = \{y \notin \mathcal{C}^* : y \in \vec{\mathcal{N}}_z^*$ for some $z \in \mathcal{C}^*\}$ (with the convention that $\tilde{\partial}\emptyset = \emptyset$) and make

$$\sigma = \sigma(p) := \mathbb{E}_p(X|\xi(x_0) = \overline{0}). \tag{4.1}$$

We also define

$$\rho = \rho(p) := \mathbb{E}_p(|\bar{\partial}\vec{\mathcal{C}}|). \tag{4.2}$$

Lemma 4.1 *For $0 \le p \le \vec{p}_c$, we have*

$$\sigma = \binom{b}{\theta - 1} q \vec{q}_\infty^{\,b-\theta} \vec{p}_\infty^{\,\theta-2}\{(b - \theta + 1)\vec{p}_\infty + (\theta - 1)\rho\}. \tag{4.3}$$

σ is analytic on $[0, \vec{p}_c)$ and

$$\sigma(p) \to \begin{cases} \infty & \textit{as } p \uparrow \vec{p}_c, \\ 0 & \textit{as } p \downarrow 0. \end{cases} \tag{4.4}$$

We now construct $\mathcal{C}$ as cluster of clusters of branching processes, much as in Sect. 3. We do that iteratively, as follows. Let $\hat{C}_0 = \mathcal{C}^*$, $S_0 = \bar{\partial}\mathcal{C}^*$, and for $n \ge 0$ make

$$\hat{C}_{n+1} = \bigcup_{y \in S_n} \mathcal{C}_y^+, \qquad S_{n+1} = \bar{\partial}\hat{C}_{n+1} = \bigcup_{y \in S_n} \bar{\partial}\mathcal{C}_y^+; \qquad \hat{\mathcal{C}} = \bigcup_{n \ge 0} \hat{C}_n. \tag{4.5}$$

When $p \le \vec{p}_c$, almost surely at each step we have the addition of a finite set (which may be empty at a given step; in this case all the subsequent additions are empty, so one may understand that the iteration stops).

Lemma 4.2 *When $p \le \vec{p}_c$, $(S_n)_{n \ge 0}$ are the successive generations of a branching process with initial distribution given by the distribution of $\tilde{X}$, and offspring distribution given by the conditional distribution of X given $\xi(x_0) = \overline{0}$.*

One may readily check that, almost surely, S survives iff $|\hat{\mathcal{C}}| = \infty$.

Lemma 4.3

$$\mathcal{C} = \hat{\mathcal{C}} \tag{4.6}$$

We now state the main results of this section, from which Theorem 1.1 follows.

Proposition 4.4 *We have*

1.

$$p_c = \inf\{p \in [0, p_f] : \sigma(p) = 1\} \in (0, \vec{p}_c); \tag{4.7}$$

2. *for $p < p_c$ there exist positive finite constants c_1, c_2 such that for all $k \ge 0$*

$$\mathbb{P}_p(|\mathcal{C}| > k) \le c_1 e^{-c_2 k}; \tag{4.8}$$

3. *at $p = p_c$, $\mathbb{P}_p(|\mathcal{C}| > k) \to 0$ as $k \to \infty$ and $\mathbb{E}_p(|\mathcal{C}|) = \infty$;*

4. *for $p > p_c$ there exist positive finite constants c_1', c_2' such that for all $k \geq 0$*

$$\mathbb{P}_p(k < |\mathcal{C}| < \infty) \leq c_1' e^{-c_2' k}. \tag{4.9}$$

Proposition 4.5

1. *π is analytic on (p_c, p_f) and left continuous at p_f.*
2. *π admits an analytic continuation from the right on p_c.*

Remark 4.6 Theorem 1.1 follows from Lemma 3.5 and Propositions 3.9, 4.4 and 4.5.

We next present proofs to the above statements. Theorem 1.2 and Proposition 4.5 will be proved in the Appendix.

Proof of Lemma 4.1 Consider the events

$$A = \left\{ \sum_{x \in \vec{\mathcal{N}}_{x_0}} 1\{\xi(x) = \underline{0} \text{ or } 1\} = \theta - 1 \right\},$$

$$B = \left\{ \sum_{x \in \vec{\mathcal{N}}_{x_0}} 1\{\xi(x) = \underline{0} \text{ or } 1\} \leq \theta - 1 \right\},$$

$$\begin{aligned}\tilde{A} = &\left\{ \sum_{i=1}^{\theta-1} 1\{\xi(x_i) = \underline{0} \text{ or } 1\} = \theta - 1 \right\} \\ &\cap \left\{ \sum_{i=\theta}^{b} 1\{\xi(x_i) = \underline{0} \text{ or } 1\} = 0 \right\},\end{aligned}$$

where $\{x_1, \ldots, x_b\}$ is an arbitrary deterministic ordering of $\vec{\mathcal{N}}_{x_0}$.

Note that $\{\xi(x_0) = \overline{0}\} = \{\eta_0(x_0) = 0\} \cap B$. Thus, since $X = 0$ in $\{\xi(x_0) = \overline{0}, A^c\}$ and $A \subset B$,

$$\begin{aligned}
\vec{q}_\infty \sigma &= \mathbb{E}_p(X, \xi(x_0) = \overline{0}) = \mathbb{E}_p(X, \xi(x_0) = \overline{0}, A) \\
&= \mathbb{E}_p(X, \eta_0(x_0) = 0, A) = \binom{b}{\theta - 1} \mathbb{E}_p(X, \eta_0(x_0) = 0, \tilde{A}) \\
&= \binom{b}{\theta - 1} \big\{ (\theta - 1)\mathbb{E}_p(|\bar{\partial}\vec{\mathcal{C}}_{x_1}|, \eta_0(x_0) = 0, \tilde{A}) \\
&\quad + (b - \theta + 1)\mathbb{P}_p(\eta_0(x_0) = 0, \tilde{A}) \big\} \\
&= \binom{b}{\theta - 1} \left\{ (\theta - 1) q \vec{p}_\infty^{\,\theta-2} \vec{q}_\infty^{\,b-\theta+1} \rho + (b - \theta + 1) q \vec{p}_\infty^{\,\theta-1} \vec{q}_\infty^{\,b-\theta+1} \right\} \\
&= \binom{b}{\theta - 1} q \vec{p}_\infty^{\,\theta-2} \vec{q}_\infty^{\,b-\theta+1} \{ (\theta - 1)\rho + (b - \theta + 1)\vec{p}_\infty \},
\end{aligned} \tag{4.10}$$

and (4.3) is verified.

The analyticity of σ follows from that of ρ, which in turn is the object of Lemma A.6 in the Appendix.

To establish (4.4) we recall that $\vec{p}_\infty$ vanishes as $p \to 0$ (see Proposition 2.1 and Remark 2.8), and so does ρ, by Lemma A.6 and the fact that $\rho(0) = 0$. This settles the second assertion of (4.4). The divergence as $p \to \vec{p}_c$ follows from the same behavior of ρ, and this is implied again by Lemma A.6 and the fact that $\rho(\vec{p}_c) = \infty$; this in turn may be seen to follow from $\mathbb{E}_{\vec{p}_c}(|\vec{\mathcal{C}}|) = \infty$ and the inequality $|\bar{\partial}\vec{\mathcal{C}}| \geq \frac{1}{2}|\vec{\mathcal{C}}|$, which in turn may be seen to follow from (A.7), if we observe that, in the context of Lemma A.4, $|\bar{\partial} C| \geq |\partial^* C|$. (That $\mathbb{E}_{\vec{p}_c}(|\vec{\mathcal{C}}|) = \infty$ follows from the divergence of the expected number of generations of the O-branching process at criticality; to see that, we may apply (I.10.8) of [8]; to check that the condition for the validity of that formula is satisfied, note that the offspring distribution of the O-branching process has an exponentially decaying tail: this follows from Lemma 3.8. See Remark 3.2.) □

Proof of Lemma 4.2 That S_0 conforms to the statement is clear. Given that the statement holds for $S_0, \ldots, S_n$, $n \geq 0$, we note that S_{n+1} depends only on η_0 restricted to S_n and on $\bigcup_{y \in S_n} \mathbb{V}_b^{*,+,y}$, which is a disjoint union. Since $\xi \equiv \overline{0}$ on S_n, the result follows. □

Proof of Lemma 4.3 That $\hat{C}_0 = \mathcal{C}^* \subset \mathcal{C}$ is clear: if $\hat{C}_0$ is nonempty, it is also internally spanned (that is, the (unoriented) bootstrapping dynamics restricted to $\hat{C}_0$ eventually fully occupies it). Assuming that $\bigcup_{i=0}^{n} \hat{C}_i \subset \mathcal{C}$, $n \geq 0$, we conclude that $\hat{C}_{n+1}$, if nonempty, becomes eventually fully occupied under the bootstrapping dynamics restricted to $\bigcup_{i=0}^{n+1} \hat{C}_i$. Since the latter set is connected and contains R, we conclude that it belongs to $\mathcal{C}$. Since n is arbitrary, we conclude that $\hat{\mathcal{C}} \subset \mathcal{C}$.

To argue the converse inclusion, we consider the following further classification of vertices of $\mathbb{V}_b^* \setminus \{R\}$. For $x \neq R$, set

$$\xi^*(x) = \begin{cases} 1, & \text{if } \eta_0^*(x) = 1, \\ \underline{0}^*, & \text{if } \eta_0^*(x) = 0 \text{ but } \eta_\infty^*(x) = 1, \\ \overline{0}^*, & \text{if } \eta_\infty^*(x) = 0, \end{cases} \tag{4.11}$$

and, making $\kappa_x = \sum_{y \in \vec{\mathcal{N}}_x^*} 1\{\xi^*(y) = 1 \text{ or } \underline{0}^*\}$, set

$$\tilde{\xi}(x) = \begin{cases} \xi^*(x), & \text{if } \xi^*(x) = 1 \text{ or } \underline{0}^*, \\ \underline{\overline{0}}, & \text{if } \xi(x) = \overline{0}^* \text{ and } \kappa(x) = \theta - 1, \\ \overline{\overline{0}}, & \text{if } \xi(x) = \overline{0}^* \text{ and } \kappa(x) < \theta - 1. \end{cases} \tag{4.12}$$

We then have that

$$\eta_\infty(x) = 0 \quad \text{if } \tilde{\xi}(x) = \overline{\overline{0}} \tag{4.13}$$

and for $x \neq R$

$$\eta_\infty(x) = 1 \quad \text{if } \tilde{\xi}(x) = \underline{\overline{0}} \text{ and } \eta_\infty(x^{*,-}) = 1. \tag{4.14}$$

Consider now a vertex x of $\mathcal{C} \neq \emptyset$ and the self avoiding path $R = x_0, x_1, \ldots, x_n = x$ connecting it to R (with $x_i \in \vec{\mathcal{N}}_{x_{i-1}}^*$ for $i = 1, \ldots, n$).

We claim that $x_i \in \hat{\mathcal{C}}$ for $i = 0, 1, \ldots, n$. To argue that, we use induction on i. An argument like the one in the last paragraph of the proof of Proposition 2.5 shows that

$\eta_\infty(R) = \eta_\infty^*(R)$, and the claim follows for $i = 0$. Suppose that it is not true for some $0 < i \le n$, and let $i_0 = \min\{i = 1, \dots, n : x_i \notin \hat{\mathcal{C}}\}$. Then we must have $\tilde{\xi}(x_{i_0}) = \bar{\bar{0}}$ (otherwise, $x_{i_0} \in \hat{\mathcal{C}}$, since, by the definition of i_0, $x_{i_0}^{*,-} \in \hat{\mathcal{C}}$, and thus $\eta_\infty(x_{i_0}^{*,-}) = 1$, and since we have (4.14)); but this contradicts (4.13). The claim is thus established, so $x \in \hat{\mathcal{C}}$, and $\mathcal{C} \subset \hat{\mathcal{C}}$. □

Proof of Proposition 4.4 Statements 1 to 3 are proved analogously as the corresponding results for $\vec{\mathcal{C}}$ in Proposition 3.9. We may follow the proof of that result, replacing $b\nu$ by σ, $\vec{\mathcal{C}}$ by $\mathcal{C}$, Remark 3.2 by the above construction of $\mathcal{C}$ as a cluster of clusters of branching processes, O_0 by $\tilde{X}$, and the exponential decay of the distribution of $\mathcal{O}$ by those of the ones of X and $\tilde{X}$.

We argue statement 4. In case $\mathbb{P}_p(\eta_\infty \equiv 1) = 1$, the claim is obvious, and so we will suppose that this is not the case. By the now familiar inequalities relating volume and perimeter of connected finite sets, we can replace $\mathcal{C}$ by Z the total family size of the S branching process. Therefore, it is enough to argue that

$$\mathbb{P}_p(k < Z < \infty) \le c_1'' e^{-c_2'' k}, \tag{4.15}$$

for some positive finite c_1'', c_2''.

In order to prove (4.15), we first observe from the proof of statements 1 to 3 above, we know that when $p_c < p \le \vec{p}_c$, $(|S_n|)_{n\ge 0}$ is a supercritical branching process started from $\tilde{X}$. Therefore, we can write

$$|S_n| = V_n^{(1)} + \cdots + V_n^{(\tilde{X})}, \tag{4.16}$$

where $V_n^{(1)}, V_n^{(2)}, \dots$ are i.i.d. copies of a branching process $(V_n)_{n\ge 0}$ starting from 1. The latter process has extinction probability

$$\gamma = \gamma(p) < 1. \tag{4.17}$$

When $p > \vec{p}_c$, S_{n+1} is still well defined by (4.5), provide $|S_n| < \infty$, but $|S_n|$ may take the value ∞ for some n. By assuming that ∞ is a trap for $(|S_n|)_{n\ge 0}$, we observe that $(|S_n|)_{n\ge 0}$ is also a branching process started from $\tilde{X}$ in this case, and (4.16) and (4.17) hold on $\{\tilde{X} < \infty\}$. It is clear that since we are supposing $\mathbb{P}_p(\eta_\infty \equiv 1) = 0$, then also $\gamma \ge \mathbb{P}_p(V_1 = 0) > 0$.

Let

$$\varphi_p(s) = \sum_{k=0}^{\infty} \mathbb{P}_p(|S_{n+1}| = k \,|\, |S_n| = 1) s^k, \quad s \ge 0$$

(which is independent of $n \ge 0$) be the probability generating function (pgf) of the offspring distribution of $(|S_n|)_{n\ge 0}$ and $(V_n)_{n\ge 0}$. Obviously

$$\varphi_p(s) < 1 \quad \text{for } 0 \le s < 1 \tag{4.18}$$

(this is true even at $s = 1$ when $|S_n|$ assumes the value ∞ with positive probability).

Now the event $\{\sum_{n=0}^{\infty} S_n = Z < \infty\}$ is precisely the event that $(|S_n|)_{n\ge 0}$ becomes extinct. Conditioning on $\{Z < \infty\}$ therefore transforms $(|S_n|)_{n\ge 0}$ into a subcritical branching process. This fact is proved as Theorem 2.1.8 in [5], where the pgf of the corresponding offspring distribution is found to be

$$\bar{\varphi}_p(s) = \sum_{k=0}^{\infty} \mathbb{P}_p(|S_{n+1}| = k \,|\, |S_n| = 1, Z < \infty) s^k = \frac{\varphi_p(s\gamma)}{\gamma}, \quad s \ge 0$$

(indep. of $n \ge 0$).

From (4.17) and (4.18), we conclude that

$$\bar{\varphi}_p(u) < \infty \quad \text{for some } u > 1.$$

The claim (4.15) now follows from Remark A.3 in the Appendix, since also

$$\mathbb{P}_p(\tilde{X} = k | Z < \infty) = \frac{\mathbb{P}_p(\tilde{X} = k)\gamma^k}{\mathbb{P}_p(Z < \infty)},$$

implying that the tail of the conditional distribution of $\tilde{X}$ given $Z < \infty$ decays exponentially. □

Remark 4.7 Note that for $x \neq R$, we have that $\tilde{\xi}(x) = \bar{\bar{0}}$ iff $\eta_0(x) = 0$ and $\sum_{y \in \vec{\mathcal{N}}_x^*} 1\{\xi^*(y) = \bar{0}^*\} \geq b - \theta + 2$. Letting $\tilde{q}_\infty = \mathbb{P}_p(\tilde{\xi}(x) = \bar{\bar{0}})$, $x \neq R$, we then have by the independence of the above random variables

$$\tilde{q}_\infty = q \sum_{i=b-\theta+2}^{b} \binom{b}{i} \vec{q}_\infty^{\,i} (1 - \vec{q}_\infty)^{b-i}. \tag{4.19}$$

In particular, $\tilde{q}_\infty$ is continuous on $[0, p_f]$ and analytic in $[0, p_f)$ as a function of p, and it is continuous at p_f.

Appendix

Proposition A.1 *Suppose that a subcritical branching process starting with a single individual is such that its offspring distribution has an exponentially decaying tail. Then the distribution of the total size of the family also has an exponentially decaying tail.*

Remark A.2 A detailed result concerning the distribution of the total size of the family, from which the above result could perhaps follow, is derived in [10] (see also Theorem I.13.1 in [8]). We could not verify a condition therein for the present case, and were thus prompted to write the proof below.

Proof of Proposition A.1 Let $(Z_n)_{n \geq 0}$ be the sizes of the successive generations of the branching process ($Z_0 = 1$), and let F_n denote the probability generating function (*pgf*) of $Z_0 + \cdots + Z_n$; and F, that of $Z_0 + Z_1 + \cdots$. We will show that for some $s > 1$

$$F(s) < \infty. \tag{A.1}$$

Let φ be the pgf of the offspring distribution. It is known ([8], Sect. I.13.2) that for $n \geq 1$ and $s \geq 0$

$$F_n(s) = s\varphi\left(F_{n-1}(s)\right). \tag{A.2}$$

From the hypotheses it follows that $\varphi(s) < \infty$ for some $s > 1$ and $\varphi'(1) < 1$, where φ' is the derivative of φ. It follows that there exist $1 < s_1 < s_0$ such that

$$s_0\varphi'(s_0) =: \alpha < 1 \quad \text{and} \quad s_1\{1 + [\varphi(s_1) - 1]/(1 - \alpha)\} \leq s_0 \tag{A.3}$$

We will argue by induction that for all $n \geq 0$

$$F_n(s_1) \leq s_0. \tag{A.4}$$

This is obvious for $n = 0$. Suppose it holds for $n \leq k$. Then, applying (A.2),

$$\begin{aligned} F_{n+1}(s_1) - F_n(s_1) &= s_1 \left[\varphi(F_n(s_1)) - \varphi(F_{n-1}(s_1))\right] \\ &\leq s_1\varphi'(F_n(s_1)) \left[F_n(s_1) - F_{n-1}(s_1)\right] \\ &\leq [s_1\varphi'(F_n(s_1))]^n \left[F_1(s_1) - F_0(s_1)\right] \\ &= [s_1\varphi'(F_n(s_1))]^n s_1[\varphi(s_1) - 1] \leq \alpha^n s_1[\varphi(s_1) - 1], \end{aligned} \tag{A.5}$$

where in the first two inequalities we have used the monotonicity of φ' and $F_\cdot(s_1)$; and in the last inequality, the induction hypothesis. Now, summing up (A.5) in $\{0, \dots, k\}$:

$$F_{k+1}(s_1) \leq F_0(s_1) + s_1[\varphi(s_1) - 1]/(1 - \alpha) = s_1 \{1 + [\varphi(s_1) - 1]/(1 - \alpha)\} \leq s_0$$

by (A.3). Equation (A.1) with $s = s_1$ follows by taking the limit as $n \to \infty$ in (A.4). □

Remark A.3 Proposition A.1 extends to the case where the initial distribution has an exponentially decaying tail. Indeed, letting ψ be the pgf of the initial distribution and $\mathcal{F}$ the pgf of the family size, then $\mathcal{F} = \psi \circ F$, where F is the pgf for the family size starting with a single individual. Supposing ψ and F are finite at some $s_0 > 1$, then by the continuity of F in $(0, s_0]$, and the fact $F(1) = 1$, we can find $s_1 > 1$ such that $F(s_1) \leq s_0$, and thus $\mathcal{F}(s_1) = \psi \circ F(s_1) < \infty$.

Lemma A.4 *For $2 \leq \theta \leq b$ and C a finite nonempty connected subset of $\mathbb{T}_b^+$ containing R, let*

$$\partial^* C = \{x \in C : |\vec{\mathcal{N}}_x \setminus C| > b - \theta\}. \tag{A.6}$$

Then

$$|\partial^* C| \geq \frac{|C|}{2}. \tag{A.7}$$

Remark A.5 It follows from the definitions of $\mathcal{W}$ and $\mathcal{O}$ and the bootstrapping dynamics that for every vertex of $\partial^*\mathcal{W}$, there is a distinct vertex in $\mathcal{O}$. It follows from Lemma A.4 that $|\mathcal{W}| \leq 2|\mathcal{O}|$ and $|\vec{\mathcal{C}}| \leq 2Z$ (see proof of Proposition 3.9).

Proof of Lemma A.4 By induction on $n = |C|$. It is true for $n = 1$ and 2 since in these cases $\partial^* C = C$ (since $\theta \geq 2$). If $|C| = n + 1 \geq 3$, then consider

$$\mathcal{M} = \{x \in \partial^* C : x \text{ is at maximal distance from the root}\}$$

and choose $x \in \mathcal{M}$ and consider x^-.

If $x^- \in \partial^* C$, then $x^- \in \partial^* (C \setminus \{x\})$, and

$$|\partial^* C| = |\partial^* (C \setminus \{x\})| + 1 \geq \frac{n}{2} + 1 \geq \frac{n+1}{2}. \tag{A.8}$$

If $x^- \notin \partial^* C$ and $x^- \notin \partial^* (C \setminus \{x\})$, then (A.8) holds again.

If $x^- \notin \partial^* C$ and $x^- \in \partial^*(C \setminus \{x\})$, then (since $\theta \geq 2$) there exists $y \in \vec{\mathcal{N}}_{x^-} \cap \mathcal{M}$, $y \neq x$, and so $x^- \in \partial^*(C \setminus \{x, y\})$. Then

$$|\partial^* C| = |\partial^*(C \setminus \{x, z\})| + 1 \geq \frac{n-1}{2} + 1 \geq \frac{n+1}{2}. \tag{A.9}$$

□

Lemma A.6 *ρ is analytic on* $[0, \vec{p}_c)$.

Proof We follow a standard approach (see Proof of Theorem 5.108 in [7]). Let $\vec{\mathcal{C}}^{(0)}$ and $\vec{\mathcal{C}}^{(1)}$ denote the vertices of $\vec{\mathcal{C}}$ which are initially 0 and 1, respectively. We write for $1 \leq n < \infty$

$$\mathbb{P}_p(|\bar{\partial}\vec{\mathcal{C}}| = n) = \sum_{m,\ell} \mathbb{P}_p(|\vec{\mathcal{C}}^{(0)}| = m, |\vec{\mathcal{C}}^{(1)}| = \ell, |\bar{\partial}\vec{\mathcal{C}}| = n) = \sum_{m,\ell} a_{nm\ell}\, p^\ell q^m \vec{q}_\infty^{\,n}, \tag{A.10}$$

where $a_{nm\ell}$ is the number of tree animals with $m + \ell$ vertices and n boundary vertices, and *orientedly spanning* configurations with m 0's and ℓ 1's in them (where by a orientedly spanning configuration of 0's and 1's in a tree animal, we mean a configuration for which that animal is internally spanned by the oriented bootstrapping dynamics). From the fact that $|\vec{\mathcal{C}}^{(0)}| + |\vec{\mathcal{C}}^{(1)}| = |\vec{\mathcal{C}}|$, and the bounds $\frac{1}{2}|\vec{\mathcal{C}}| \leq |\bar{\partial}\vec{\mathcal{C}}| \leq b|\vec{\mathcal{C}}|$, the right hand side of (A.10) is bounded below by $r^{2n} \sum_{m,\ell} a_{nm\ell}$, where $r = p \wedge q \wedge \sqrt{\vec{q}_\infty}$, and since the left hand side of that equation is a probability, and the latter sum is independent of p, we get

$$\sum_{m,\ell} a_{nm\ell} \leq \lambda^n, \tag{A.11}$$

with $\lambda = \inf_{0<p<p_f} r^{-2} < \infty$, since $\vec{q}_\infty > 0$ for $p < p_f$, and $p_f > 0$.

We now claim that in the sum on the right hand side of (A.10), if $a_{nm\ell} > 0$, then there exists a constant $c > 0$ such that $\ell \geq cn$. Indeed, let $W_1, \dots, W_k$ be the $\underline{0}$-subclusters of $\vec{\mathcal{C}}$, and $O_1, \dots, O_k$ their 1-boundaries. Then one readily checks that

$$|\vec{\mathcal{C}}^{(0)}| + |\vec{\mathcal{C}}^{(1)}| = |\vec{\mathcal{C}}| \geq \frac{1}{b}|\bar{\partial}\vec{\mathcal{C}}|, \tag{A.12}$$

$$|\vec{\mathcal{C}}^{(1)}| \geq |O_1| + \cdots + |O_k| \geq \frac{1}{2}(|W_1| + \cdots + |W_k|) = \frac{1}{2}|\vec{\mathcal{C}}^{(0)}|, \tag{A.13}$$

where the second inequality in (A.13) follows by an application of Lemma A.4, as in the proof of Proposition 3.9. The claim follows with $c = 1/3b$.

It also follows from one of the above inequalities ($\frac{1}{2}|\vec{\mathcal{C}}| \leq |\bar{\partial}\vec{\mathcal{C}}|$), that

$$m \vee \ell \leq 2n. \tag{A.14}$$

Now using the fact that $\vec{q}_\infty$ is analytic in (a complex neighborhood of) $[0, p_f)$, we write, formally at this point, in a complex neighborhood of $[0, \vec{p}_c)$,

$$\rho(z) = \sum_{n\geq 1} n \sum_{m,\ell} a_{nm\ell} z^\ell (1-z)^m \vec{q}_\infty(z)^n. \tag{A.15}$$

I) To get analyticity of ρ at $[0, \vec{p}_c)$, it is enough to check that the series in (A.15) has a uniformly convergent tail at a neighborhood of any point of that interval. From the above discussion, we have

$$\left|\sum_{m,\ell} a_{nm\ell} z^\ell (1-z)^m \vec{q}_\infty(z)^n\right| \le \sum_{m,\ell} a_{nm\ell} |z|^{cn} (1+|z|)^{2n} 2^n$$
$$= |z|^{cn}(1+|z|)^{2n}2^n \sum_{m,\ell} a_{nm\ell} \le (d(z))^n$$

at a neighborhood of the origin where $|\vec{q}_\infty(z)| \le 2$ (which exists by continuity of ρ at the origin, where it equals 1), and where $d(z) = 2\lambda|z|^c(1+|z|)^2 < 1$. The uniform convergence of the series tail around the origin follows, and ρ is analytic at the origin.

II) Given $p \in (0, \vec{p}_c)$ and $\delta > 0$ such that $|z-p| \le \delta$ and $|\vec{q}_\infty(z) - \vec{q}_\infty(p)| \le \delta$, we have that

$$\left|\sum_{m,\ell} a_{nm\ell} z^\ell (1-z)^m \vec{q}_\infty(z)^n\right|$$
$$\le \sum_{m,\ell} a_{nm\ell} p^\ell q^m \vec{q}_\infty(p)^n \left(\frac{p+\delta}{p}\right)^\ell \left(\frac{q+\delta}{q}\right)^m \left(\frac{\vec{q}_\infty(p)+\delta}{\vec{q}_\infty(p)}\right)^n$$
$$\le c(p,\delta)^n \sum_{m,\ell} a_{nm\ell} p^\ell q^m \vec{q}_\infty(p)^n = c(p,\delta)^n \mathbb{P}_p(|\bar{\partial}\vec{\mathcal{C}}| = n) \le d(p,\delta)^n,$$

where $c(p,\delta) = (\frac{(p+\delta)(q+\delta)}{pq})^2 (\frac{\vec{q}_\infty(p)+\delta}{\vec{q}_\infty(p)}) \to 1$ as $\delta \to 0$, and where the second inequality follows from (A.14), and thus, by the exponential decay of the tail of the distribution of $|\bar{\partial}\vec{\mathcal{C}}|$ (which follows from that of $|\vec{\mathcal{C}}|$ and $|\bar{\partial}\vec{\mathcal{C}}| \le b|\vec{\mathcal{C}}|$), $d(p,\delta)$ can be taken strictly less than 1 by making δ sufficiently small. The uniform convergence of the tail of the series in (A.15) on $|z-p| \le \delta$ follows, and the argument is complete. □

Proof of Proposition 4.5

1. An entirely similar argument as in II) of the proof of Lemma A.6 can be made starting from

$$1 - \pi(p) = q_\infty + \mathbb{P}_p(0 < |\mathcal{C}| < \infty), \tag{A.16}$$

where $q_\infty = 1 - p_\infty$. Since p_∞ is analytic in $[0, p_f)$, it remains to consider $\mathbb{P}_p$ $(0 < |\mathcal{C}| < \infty)$, which can be written as follows.

$$\sum_{n\ge1}\sum_{k,\ell,m} \mathbb{P}_p(|\mathcal{C}^{(0)}| = m, |\mathcal{C}^{(1)}| = \ell, |\bar{\partial}\mathcal{C}| = k) = \sum_{n\ge1}\sum_{k,\ell,m} \tilde{a}_{nkm\ell} p^\ell q^m \tilde{q}_\infty^k, \tag{A.17}$$

where $\mathcal{C}^{(0)}$ and $\mathcal{C}^{(1)}$ denote the vertices of $\mathcal{C}$ which are initially 0 and 1, respectively; $\ell + m = n$; $\tilde{a}_{nkm\ell}$ is the number of tree animals with $m+\ell$ vertices and k boundary vertices, and *spanning* configurations with m 0's and ℓ 1's in them (where by a spanning configuration of 0's and 1's in a tree animal, we mean a configuration for which that animal is internally spanned by the unoriented bootstrapping dynamics); and then using the bound $k \le bn$, the analyticity of $\tilde{q}_\infty$ and the exponential decay of $\mathbb{P}_p(|\mathcal{C}| = n)$ as $n \to \infty$ for $p > p_c$, which follows from (4.9).

This argument works for getting the analyticity result. It works as well for getting left continuity at p_f when $\theta < b$, since in this case $\tilde{q}_\infty(p_f) > 0$. When $\theta = b$, since in this case $\tilde{q}_\infty(p_f) = 0$, one needs instead to argue like in I) of the proof of Lemma A.6. See the proof of Theorem 1.2 below, where the discussion is done in some more detail.

2. Consider the branching process starting with a single individual and with offspring distribution given by the conditional distribution of X given that $\xi(R) = \overline{0}$. Its extinction probability $s = d(p)$ satisfies the equation

$$s = \varphi^+(s) = \mathbb{E}_p(s^X | \xi(R) = \overline{0}) \tag{A.18}$$

(see paragraph above (4.1)). Arguing as in the proof of Lemma 4.1 we get

$$\begin{aligned}
\vec{q}_\infty \varphi^+(s) &= \binom{b}{\theta-1} \mathbb{E}_p[s^X, \eta_0(R) = 0, \tilde{A}] + q\mathbb{P}_p(B \setminus A) \\
&= q\binom{b}{\theta-1} s^{b-\theta+1} \{\mathbb{E}_p[s^Y; Y \geq 1]\}^{\theta-1} \vec{q}_\infty^{\,b-\theta+1} + q\sum_{i=0}^{\theta-2} \binom{b}{i} \vec{p}_\infty^{\,i} \vec{q}_\infty^{\,b-i} \\
&= q\binom{b}{\theta-1} \vec{p}_\infty^{\,\theta-1} \vec{q}_\infty^{\,b-\theta+1} s^{b-\theta+1} \{\mathbb{E}_p[s^Y | Y \geq 1]\}^{\theta-1} \\
&\quad + q\sum_{i=0}^{\theta-1} \binom{b}{i} \vec{p}_\infty^{\,i} \vec{q}_\infty^{\,b-i} - q\binom{b}{\theta-1} \vec{p}_\infty^{\,\theta-1} \vec{q}_\infty^{\,b-\theta+1} \\
&= q\binom{b}{\theta-1} \vec{p}_\infty^{\,\theta-1} \vec{q}_\infty^{\,b-\theta+1} (s^{b-\theta+1} \{\mathbb{E}_p[s^Y | Y \geq 1]\}^{\theta-1} - 1) + \vec{q}_\infty,
\end{aligned} \tag{A.19}$$

where $Y = |\bar{\partial}\vec{\mathcal{C}}|$, and the identity $\vec{q}_\infty = q\sum_{i=0}^{\theta-1} \binom{b}{i} \vec{p}_\infty^{\,i} \vec{q}_\infty^{\,b-i}$ used in the last equality above follows from Proposition 2.1. After a straightforward calculation, one gets that (A.18) is equivalent to

$$(1-s)G(p) = 1 - s^{b-\theta+1} \{\mathbb{E}_p[s^Y | Y \geq 1]\}^{\theta-1} = 1 - s^b \{\mathbb{E}_p[s^{Y-1} | Y \geq 1]\}^{\theta-1}, \tag{A.20}$$

where $G(p) = 1/\binom{b}{\theta-1} \vec{p}_\infty^{\,\theta-1} \vec{q}_\infty^{\,b-\theta}$. We rewrite the right hand side of (A.20) as

$$\begin{aligned}
&1 - s^b + s^b(1 - \{\mathbb{E}_p[s^{Y-1} | Y \geq 1]\}^{\theta-1}) \\
&\quad = (1-s)\sum_{i=0}^{b-1} s^i + s^b[1 - \mathbb{E}_p(s^{Y-1} | Y \geq 1)] \sum_{i=0}^{\theta-2} \left\{\mathbb{E}_p\left(s^{Y-1} \middle| Y \geq 1\right)\right\}^i.
\end{aligned} \tag{A.21}$$

Now the expression in square brackets in the right hand side of (A.21) can be expressed as

$$\frac{1-s}{\vec{p}_\infty} \mathbb{E}_p\left(\sum_{i=0}^{Y-2} s^i; Y \geq 2\right) \tag{A.22}$$

Substituting (A.21) and (A.22) in the right hand side of (A.20), we find that it can be reexpressed as $1-s$ times

$$\begin{aligned}
I(p,s) &:= \sum_{i=0}^{b-1} s^i + \frac{s^b}{\vec{p}_\infty} \mathbb{E}_p\left(\sum_{i=0}^{Y-2} s^i; Y \geq 2\right) \sum_{i=0}^{\theta-2} \left\{\mathbb{E}_p\left(s^{Y-1} \middle| Y \geq 1\right)\right\}^i \\
&=: \sum_{i=0}^{b-1} s^i + \frac{s^b}{\vec{p}_\infty} I_1(p,s) \sum_{i=0}^{\theta-2} I_2(p,s)^i.
\end{aligned} \tag{A.23}$$

By Lemma A.7 below, we have that I_1 and I_2 are analytic in $(0, p_c + \epsilon) \times (0, 1 + \epsilon)$ for $\epsilon > 0$ small enough. Then so is I.

Let now $s = e(p)$ be the solution of $I(p, s) = G(p)$ in a neighborhood of p_c. We have that

$$e(p) = d(p) \quad \text{for } p \geq p_c, \tag{A.24}$$

and thus $e(p_c) = 1$. One also readily checks that $\frac{d}{ds} I(p, s) > 0$ in $(0, \vec{p}_c) \times (0, 1 + \epsilon)$. We may then apply the Analytic Implicit Function Theorem to conclude that e is well defined and analytic in a neighborhood of p_c. We then have from (A.24) that e is the analytic continuation of d on p_c.

We will argue now that $1 - \pi = h(p, d)$ with h analytic on $(p_c, 1)$. The result then follows, with $h(p, e)$ as the analytic continuation of $1 - \pi$ at p_c.

Indeed, $1 - \pi = \sum_{n=0}^{\infty} d^n \mathbb{P}_p(\tilde{X} = n) =: h(p, d)$. Proceeding as in the proof of Lemma A.6, we expand $\mathbb{P}_p(\tilde{X} = n) = \mathbb{P}_p(|\bar{\partial}\mathcal{C}^*| = n)$ as done in (A.10):

$$\mathbb{P}_p(|\bar{\partial}\mathcal{C}^*| = n) = \sum_{m,\ell} a^*_{nm\ell} p^\ell q^m \vec{q}^{\,n}_\infty, \tag{A.25}$$

so we can express h formally as a function of two complex variables as

$$h(z, w) = \sum_{n=0}^{\infty} w^n \sum_{m,\ell} a^*_{nm\ell} z^\ell (1 - z)^m \vec{q}^{\,n}_\infty(z); \tag{A.26}$$

again, in order to establish the analyticity of h in $(p_c, 1)$, it suffices to show that the tail of the first sum in (A.26) converges uniformly in the product say $\Pi = B_1 \times B_2$ of two balls in the complex plane around p_c and 1 respectively, with radii say $\delta > 0$ small to be chosen presently. Arguing as before, we get that

$$\begin{aligned} &\left| \sum_{n=M}^{\infty} w^n \sum_{m,\ell} a^*_{nm\ell} z^\ell (1 - z)^m \vec{q}^{\,n}_\infty(z) \right| \\ &\quad \leq \sum_{n=M}^{\infty} c^n \sum_{m,\ell} a^*_{nm\ell} p_c^\ell (1 - p_c)^m \vec{q}^{\,n}_\infty(p_c) \\ &\quad = \sum_{n=M}^{\infty} c^n \mathbb{P}_{p_c}(\tilde{X} = n), \end{aligned} \tag{A.27}$$

where $c = \sup_{w \in B_2} |w| \sup_{z \in B_1} (\frac{|z(1-z)|}{p_c(1-p_c)})^2 \frac{|\vec{q}_\infty(z)|}{\vec{q}_\infty(p_c)}$. Since c can be made close to 1 by making δ close to 0, the result follows by the exponential decay of the distribution of $\tilde{X}$ below $\vec{p}_c$. □

Lemma A.7 *I_1 and I_2 defined in (A.23) above are analytic in* $(0, \bar{p}) \times (0, 1 + \epsilon)$ *for all* $\bar{p} < \vec{p}_c$ *and* $\epsilon = \epsilon(\bar{p}) > 0$ *small enough.*

Proof The latter function can be expressed as

$$I_2(p, s) = \frac{1}{\vec{p}_\infty} \sum_{n=1}^{\infty} s^n \mathbb{P}_p(Y = n) = \frac{1}{\vec{p}_\infty} \sum_{n=1}^{\infty} s^n \mathbb{P}_p(|\bar{\partial}\vec{\mathcal{C}}| = n). \tag{A.28}$$

We then replace (A.10) above and proceed similarly as in the proof of Lemma A.6, part II. (Note that at a point of the argument, we need to take $\epsilon > 0$ small enough; we again use the exponential decay of the tail of the distribution of $|\bar{\partial}\vec{\mathcal{C}}|$, which holds below $\vec{p}_c$.)

As for I_1, we reexpress it as

$$I_1(p,s) = \sum_{n=0}^{\infty} s^n \sum_{k=n+2}^{\infty} \mathbb{P}_p(|\bar{\partial}\vec{\mathcal{C}}| = n), \tag{A.29}$$

and then again proceed as in the proof of Lemma A.6. □

Proof of Theorem 1.2

1. Recalling the discussion in the first paragraph of Lemma 3.5, in this case, for $p \le p_f$, we have that $\vec{p}_\infty$ is the solution $x(p)$ of $x + x^2 + \cdots + x^{b-1} = p/q$ in $(0,1)$. Now $x(p)$ is well defined and indeed analytic on $(0,1)$ (by the Analytic Implicit Function Theorem), and of course coincides with $\vec{p}_\infty$ on $(0, p_f]$. We thus have that $x(p)$ is an analytic continuation of $\vec{p}_\infty$ on p_f, and from (2.3) and (4.19), we find analytic continuations of both q_∞ and $\tilde{q}_\infty$ on p_f, $\bar{x}(p)$ and $\tilde{x}(p)$, respectively.

To get the result for π, we consider the function

$$y(p) := \bar{x}(p) + \sum_{n\ge 1}\sum_{k,\ell,m} \tilde{a}_{nkm\ell}\, p^\ell q^m \tilde{x}(p)^k \tag{A.30}$$

(see (A.16–A.17)) and argue as in the proof of Proposition 4.5 to find that the sum on the right hand side above is uniformly convergent on a complex neighborhood of p_f. For that we use the fact that $\tilde{x}$ is continuous on a complex neighborhood of p_f and $\tilde{x}(p_f) = 0$ to get that given $\delta > 0$, we find a complex neighborhood of p_f such that $\tilde{x} \le \delta$ in that neighborhood. We also have as in (A.11) that $\sum_{k,\ell,m} \tilde{a}_{nkm\ell} \le \tilde{\lambda}^n$ for a suitable finite $\tilde{\lambda}$. We conclude that y is analytic on p_f, and it is thus an analytic continuation of $1 - \pi$ in that point.

2. We start by establishing (1.15) for π replaced by $\vec{p}_\infty$: we claim

$$p'_\infty(p) = \frac{d}{dp}\vec{p}_\infty(p) \to \infty \quad \text{as } p \uparrow p_f. \tag{A.31}$$

Indeed from Proposition 2.1 and the Implicit Function Theorem, we have that

$$p'_\infty(p) = \left.\frac{\frac{d}{dp} f_p(x)}{1 - f'_p(x)}\right|_{x=\vec{p}_\infty} = \frac{1 - \sum_{k=\theta}^{b} \binom{b}{k} \vec{p}_\infty^{\,k} (1-\vec{p}_\infty)^{b-k}}{1 - f'_p(\vec{p}_\infty)}. \tag{A.32}$$

Since $\vec{p}_\infty(p_f) < 1$, we have that the numerator of (A.32) is bounded away from zero as $p \uparrow p_f$. On the other hand, the denominator vanishes as $p \uparrow p_f$ (see Remark 2.4). Equation (A.31) follows.

Now by (A.17) and statement 5 of Theorem 1.1, we have for $p_c < p < p_f$

$$\begin{aligned}\pi'(p) &= p'_\infty(p) - \tilde{q}'_\infty(p) \sum_{n\ge 1}\sum_{k,\ell,m} k\tilde{a}_{nkm\ell}\, p^\ell q^m \tilde{q}_\infty^{k-1}(p) \\ &\quad + \sum_{n\ge 1}\sum_{k,\ell,m} \tilde{a}_{nkm\ell} (p^\ell q^m)' \tilde{q}_\infty^k(p).\end{aligned} \tag{A.33}$$

We have that p_∞ is increasing and analytic, so $p'_\infty \ge 0$. By (4.19) and (A.31), one readily gets that $\tilde{q}'_\infty(p) \to -\infty$ as $p \uparrow p_f$. It is easy to see that the factor of $\tilde{q}'_\infty(p)$ in (A.33) is bounded away from zero in a left neighborhood of p_f. So, to get the result, it is enough to show that the latter summand in (A.33) is bounded in a left neighborhood of p_f. To argue that, we take $p_f - \epsilon < p < p_f$, where $\epsilon > 0$ will be chosen small enough later on, and bound the absolute value of that term by

$$c \sum_{n\geq 1} n \sum_{k,\ell,m} \tilde{a}_{nkm\ell} p^{\ell} q^{m} \tilde{q}_{\infty}^{k}(p), \tag{A.34}$$

where c is the constant obtained by bounding above ℓ and m in the above sum in terms of n, as usual, times $p^{-1} + q^{-1}$. The above sum can now be bounded above by

$$\sum_{n\geq 1} n\delta^{n} \sum_{k,\ell,m} \tilde{a}_{nkm\ell} p_f^{\ell} (1-p_f)^{m} \tilde{q}_{\infty}^{k}(p_f) = \mathbb{E}_{p_f}(|\mathcal{C}|\delta^{|\mathcal{C}|}; |\mathcal{C}| < \infty), \tag{A.35}$$

where $\delta = (\sup_{p_f-\epsilon<p<p_f} \frac{(1-p_f)\tilde{q}_{\infty}(p_f)}{(1-p)\tilde{q}_{\infty}(p)})^{d}$, and d is the constant obtained by bounding above m and k in the above sum in terms of n, as usual. It is now clear by the continuity of $\tilde{q}_{\infty}$ that δ can be made arbitrarily close to 1 by taking ϵ small enough. The result follows from statement 4 of Theorem 1.1. □

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